The Direct Measurement of the Napierian Base

1913 Proc. Phys. Soc. London 26 178

(http://iopscience.iop.org/1478-7814/26/1/321)

XXI. *The Direct Measurement of the Napierian Base.* By ROLLO APPLEYARD, *M.Inst.C.E.*

RECEIVED MARCH 5, 1914.

THE equations of the catenary suggest that it should be possible to derive the Napierian base e from direct measurements of a suspended cord or chain. Such measurements will be found to be a useful exercise for students, especially as an introduction to the employment of hyperbolic functions in solving problems in physics. In Fig. 1, let

a=Length OV of chain, of weight equal to the tension at V, when the catenary is complete on both sides of V.

s=Length of the curved portion VP, where P is any point (x, y) on the catenary.

The familiar equations are

$$y=a\frac{e^{\frac{x}{a}}+e^{-\frac{x}{a}}}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$s=a\frac{e^{\frac{x}{a}}-e^{-\frac{x}{a}}}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, at the point F on the curve, corresponding to $x=a$, $y=y_1$, we have

$$e=\frac{y_1+s_1}{a}$$

and

$$\frac{1}{e}=\frac{y_1-s_1}{a}.$$

If a is regarded as the unit of length, e is represented in Fig. 1 by FG plus the length of chain FV when straightened out; for when $x=a=1$, we have

$$e=y_1+s_1$$

and

$$\frac{1}{e}=y_1-s_1.$$

In performing the experiment, the chief difficulty is to ascertain the tension at V without altering the conditions of equilibrium. A pulley running on ball bearings was tried at V with unsuccessful results. Accuracy demands there the equivalent of a frictionless pulley of negligible dimensions, and this requirement is fulfilled by substituting for a pulley a loop of fine thread, as shown along the line VB in Fig. 1. This loop is slipped over the end of the chain up to the point V, and the other end of it is attached to a fixed pin at B. The end D of the chain can then be moved about until VB makes an angle of 45 deg. with the vertical. Then, if V is the lowest

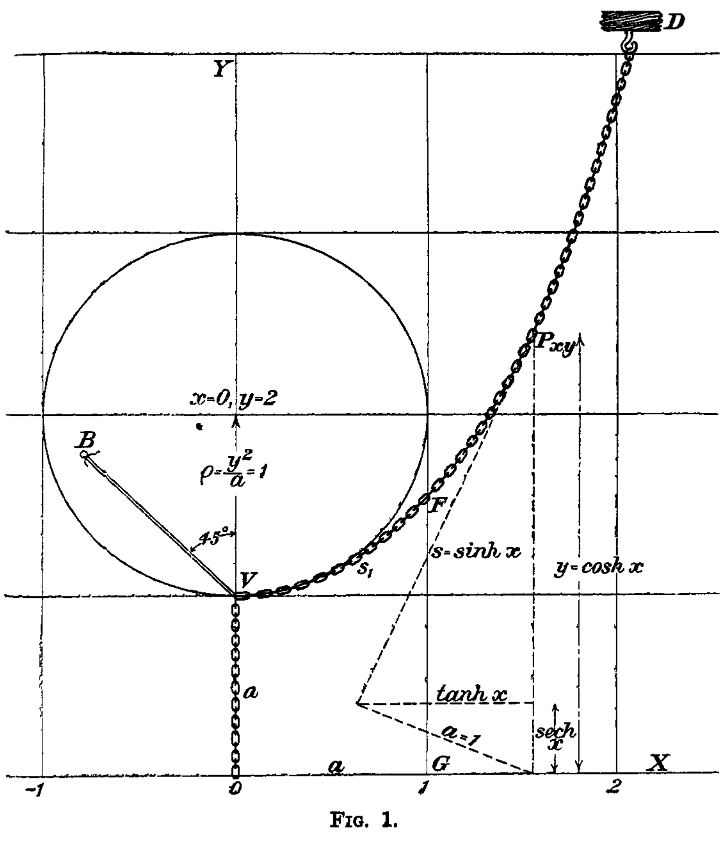

FIG. 1.

point of a catenary, the tangent at V is horizontal, and a is equal to, and at right-angles to, the tension of the catenary at V. Experiment shows however, that the end D of the chain can be moved over a wide range in various directions while retaining the thread VB unaltered in direction along the 45 deg. line, and only the roughest estimate of the value of e

can be made with the apparatus in that form. The explanation is that it is impossible to judge by eye when the tangent to the catenary at V is truly horizontal, and a small error in this estimation causes a considerable error in the measurement of y_1. To complete the method, therefore, some additional means must be found for ensuring that the curve reaches its lowest point at V, that its tangent is there horizontal, and that VB remains at 45 deg. to the vertical. This adjustment can be secured by introducing the circle of curvature, the radius of which for any value of y is

$$\rho=\frac{y^2}{a},$$

and the centre of which is always at the point $\left[\left(x-\frac{ys}{a}\right), 2y\right]$.

Hence at V, where $y=1=a$, and where $x=s=0$, the radius is 1, and the centre is at $2y=2$. After drawing the circle, the operation consists in finding the position of the end D, such that while the thread attached to the chain at V is maintained along the line VB, the curved portion of the chain, beginning at V, follows as closely as possible the circle of curvature near V. It, of course, soon departs from the circle, but these two restraints together are sufficient to allow a satisfactory estimate of e to be made. The same apparatus enables sinh x, cosh x, tanh x, and sech x to be found with a piece of chain in like manner, as represented in Fig. 1, and the student is led to see that these functions, which for so long have been forced into association with the hyperbola, are far more easily exemplified by the catenary, and that x then becomes a definite length, instead of an obscure area or an elusive angle.

Illustration of Logarithmic Series.

When e has been determined as the sum of the lengths FG and FV, in terms of the length VO as unit, a helpful exercise for the student is to consider what FG and FV separately represent. By direct measurement he finds FG=1·543, which is cosh 1; and FV=1·175, which is sinh 1. Now, if he writes down the series defining e—*i.e.*,

$$e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}\;\cdots$$

and selects and adds, first the odd terms and then the even terms, he obtains 1·543 and 1·175, as given by the chain measurements ; and he recognises here the particular case when $x=1$, of the series and relations

$$\operatorname{Cosh} x=1+\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}+\frac{x^6}{\lfloor 6}+\ \ldots$$

$$\operatorname{Sinh} x=x+\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}+\frac{x^7}{\lfloor 7}+\ \ldots$$

$$e^x=\cosh x+\sinh x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\ \ldots$$

The point P may then be supposed to move along the chain, and the corresponding changes in the hyperbolic functions of x can be traced in an instructive manner, as represented in Fig. 1.

Logarithms from Chain Measurements.

From the fact that inverse hyperbolic functions can be put into logarithmic form, it appeared likely that chain measurements could be utilised to find logarithms; and this proved to be the case. This can be shown either from equation (1) or from equation (2). Thus, equation (1) can be written

$$e^{\frac{x}{a}}=\frac{1}{a}(y+\sqrt{y^2-a^2}),$$

which gives

$$\frac{x}{a}=\log_e\frac{1}{a}+\log_e(y+\sqrt{y^2-a^2}). \quad . \quad . \quad . \quad (3)$$

Similarly, equation (2) gives

$$\frac{x}{a}=\log_e\frac{1}{a}+\log_e(s+\sqrt{s^2+a^2}). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But Fig. 1 shows that $\sqrt{y^2-a^2}=s$ and $\sqrt{s^2+a^2}=y$. Hence, from (3) or (4),

$$\frac{x}{a}=\log_e\frac{1}{a}+\log_e(s+y). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $s=\sinh\frac{x}{a}$ and $y=\cosh\frac{x}{a}$.

Now, when $a=1$,

$$x=\log_e(s+y).$$

It follows that at any point along the chain suspended as in

Fig. 1 the Napierian logarithm of any number represented by a length $(s+y)$ is the corresponding x ordinate. For example, at the point $x=2$, $s=3{\cdot}627$ and $y=3{\cdot}762$; which means that 2 is the Napierian logarithm of $3{\cdot}627+3{\cdot}762=7{\cdot}389$. The extension to the determination of common logarithms involves the modulus, and is less simple. In conclusion, it may be pointed out that the chief object of the experiment is to direct attention to the value of the means afforded by a suspended chain for enabling a number of well-known functions to be presented in a form in which they can be co-related and remembered. A physicist cast upon a desert island and bereft of all but a chain could, to a first degree of approximation, determine the Napierian base, the hyperbolic functions, and the natural logarithms, and he could perform the operations of multiplication and division by mechanical means.

ABSTRACT.

The author described a simple apparatus intended to convey to students an idea of the way in which the base e of the Napierian logarithms enters into physical problems in a specific case of wide application. A small length of chain is allowed to hang from a loop of thread, and the remaining part of the chain is then pulled aside until the thread is at 45 deg. to the vertical. The curved portion becomes a true catenary when the angle between the vertical and curved portions of chain at the attachment of the loop is 90 deg. To ensure that this condition is reached, the circle of curvature of the catenary at that point is drawn, and this is found to have a radius equal to the vertical portion. In these circumstances, if the vertical length is taken as unity, and if its lower end is taken as origin, it is shown that e is the sum of the y-ordinate at $x=1$, and the length of curved chain between the point where that y-ordinate cuts the curve and the top of the vertical portion. The application of this result to a simple representation of the relationship and meaning of hyperbolic functions was also shown, and it was urged that such functions should be studied from consideration of the catenary rather than from the hyperbola.

DISCUSSION.

The CHAIRMAN supposed the object of the Paper was to familiarise students with the properties of the catenary rather than the determination of e, as this was so easily obtained from the formula.

Dr. W. H. ECCLES drew attention to a method of measuring e from the properties of the catenary, which the late Prof. Minchin used to set as a practical problem in the London University examinations.